Fluid Mechanics
2019; 5(1): 15-25
http://www.sciencepublishinggroup.com/j/fm
doi: 10.11648/j.fm.20190501.13
ISSN: 2575-1808 (Print); ISSN: 2575-1816 (Online)

A Class of Exact Solutions for a Variable Viscosity Flow with Body Force for Moderate Peclet Number ViaVon-Mises Coordinates

Mushtaq Ahmed

Department of Mathematics, University of Karachi, Karachi, Pakistan

Email address:
mushtaqku@yahoo.com, muahmed@uok.edu.pk

To cite this article:
Mushtaq Ahmed. A Class of Exact Solutions for a Variable Viscosity Flow with Body Force for Moderate Peclet number Via Von-Mises Coordinates. *Fluid Mechanics*. Vol. 5, No. 1, 2019, pp. 15-25. doi: 10.11648/j.fm.20190501.13

Received: March 13, 2019; **Accepted**: May 7, 2019; **Published**: June 4, 2019

Abstract: The objective of this article is to communicate a class of new exact solutions of the plane equation of momentum with body force, energy and continuity for moderate Peclet number in von-Mises coordinates. Viscosity of fluid is variable but its density and thermal conductivity are constant. The class characterizes the streamlines pattern through an equation relating two continuously differentiable functions and a function of stream function ψ. Applying the successive transformation technique, the basic equations are prepared for exact solutions. It finds exact solutions for class of flows for which the function of stream function varies linearly and exponentially. The linear case shows viscosity and temperature for moderate Peclet number for two variety of velocity profile. The first velocity profile fixes both the functions of characteristic equation whereas the second keeps one of them arbitrary. The exponential case finds that the temperature distribution, due to heat generation, remains constant for all Peclet numbers except at 4 where it follows a specific formula. There are streamlines, velocity components, viscosity and temperature distribution in presence of body force for a large number of the finite Peclet number.

Keywords: Successive Transformation Technique, Variable Viscosity Fluids, Navier-Stokes Equations with Body Force, Martin's Coordinates, Von-MisesCoordinates

1. Introduction

Theoretical study of a fluid flow problem with variable viscosity is a system containing equation of momentum, energy and continuity. The momentum equations for the motion of a fluid element are the Navier-Stokes equations (NSE) having capacity to incorporate all forces in the right-hand side. In presence of unknown external force this system for steady flow with constant density, thermal conductivity and specific heat using following dimensionless parameters

$$x^* = \frac{x}{L_0} \quad y^* = \frac{y}{L_0} \quad u^* = \frac{u}{U_0} \quad v^* = \frac{v}{U_0}$$

$$\mu^* = \frac{\mu}{\mu_0} \quad p^* = \frac{p}{p_0} \quad F_1^* = \frac{F_1}{F_0} \quad F_2^* = \frac{F_2}{F_0}$$

Dropping the overhead "*" are following.

$$\nabla . \mathbf{v} = \frac{\partial v_k}{\partial x_k} = 0 \qquad (1)$$

$$\left(v_k \frac{\partial v_i}{\partial x_k} \right) = F_i - \frac{\partial p}{\partial x_i} + \frac{1}{R_e} \frac{\partial}{\partial x_j} \left\{ \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \right\} \qquad (2)$$

$$\left(v_k \frac{\partial T}{\partial x_k} \right) = \frac{1}{R_e P_r} \frac{\partial}{\partial x_i} \left(\frac{\partial T}{\partial x_i} \right) + \frac{E_c}{R_e} \mu \frac{\partial v_i}{\partial x_j} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \qquad (3)$$

Equations (1-3) are in tensor notation where velocity vector is $\mathbf{v} = v_k(x_i)$, $p = p(x_i)$ is pressure, $\mu = \mu(x_i) > 0$ is viscosity and $\mathbf{F} = F_j(x_i)$ is the body force per unit mass $i, j, k \in \{1,2,3\}$. The quantities P_r, E_c and R_e are the *Prandtl number*, the *Eckert number*, the *Reynolds number* respectively. The product of R_e and P_r is Peclet number $P_{e'}$. For the plane case, in Cartesian space (x, y), taking

$i, j, k \in \{1,2\}$, $x_1 = x$, $x_2 = y$, $\mathbf{F} = (F_1(x,y), \quad F_2(x,y))$, $v_1 = u \;\; v_2 = v$, the equations (1-3) reduce to following system of equations

$$u_x + v_y = 0 \tag{4}$$

$$u u_x + v u_y = F_1 - p_x + [(2\mu u_x)_x + \{\mu(u_y + v_x)\}_y] \tag{5}$$

$$u v_x + v v_y = F_2 - p_y + \frac{1}{\mathrm{R_e}}[(2\mu v_y)_y + \{\mu(u_y + v_x)\}_x] \tag{6}$$

$$u T_x + v T_y = \frac{1}{\mathrm{R_e}\,\mathrm{P_r}}\left(T_{xx} + T_{yy}\right) + \frac{\mathrm{E_c}}{\mathrm{R_e}}[2\mu(u_x{}^2 + v_y{}^2) + \mu(u_y + v_x)^2] \tag{7}$$

The complex mathematical structure of this system of fundamental equations (4-7) requires using new solution techniques. The solution technique like one-parameter group, Martin's system, hodograph transformation method and successive transformation methods in absence of body force are there in [1-5]. The body force term appears in the study of magneto-hydrodynamic and in geophysical fluid dynamics [6-8] for example. The exact solution of fundamental equations with body force by setting the arbitrary coordinates of the Martin's system in radial directionare there in [9-13]. Further, the solution of the basic system of equations is found for very large and very small $P_{e'}$ where as the solution for moderate $P_{e'}$ is challenging. Please refer to [14-16] and references therein.

The solution of the plane equation of continuity (5) provides astream function $\psi = \psi(x,y)$, such that $\psi_{yx} = \psi_{xy}$, and

$$\frac{\partial \psi}{\partial y} = u, \frac{\partial \psi}{\partial x} = -v \tag{8}$$

This discourse applies successive coordinate transformations technique for solution of plane momentum and energy equations (5-7). It transforms equations firstly into a curvilinear net (ϕ,ψ) where the coordinate curves $\psi = const.$ as streamlines and the coordinate curves $\phi = const.$ may take any direction. With this definition, the coordinate system (ϕ,ψ) is here Martin's coordinate system (ϕ,ψ) for it has used in Martin [17]. Secondly, it retransforms the basic equations into von-Mises coordinates. In von-Mises coordinates (x,ψ) the coordinate lines $\phi =$ constant of Martin's coordinates is taken along $x - axis$ thus the function $\phi = x$ and stream function ψ of Martin's coordinates as independent variables instead of y and x [18]. The von-Mises coordinates (x,ψ) takes the privilege of defining the arbitrary curve $\phi = const.$ of Martin's system by setting

$$\phi = x \tag{9}$$

This communication characterizes the pattern of streamlines $\psi = const.$ by

$$\frac{y - f(x)}{g(x)} = const \tag{10}$$

The equation (7) implies

$$y = f(x) + g(x)\nu(\psi) \tag{11}$$

Where $f(x)$ and $g'(x) \neq 0$ are continuously differentiable functions and $\nu(\psi)$ is a functions of stream function.

The paper is organized as follow: Section (2) successively transforms basic equations into the von-Mises coordinates (x,ψ). Section (3), provides exact solutions of fundamental equations for the cases when $\nu(\psi)$ varies linearly and when it varies exponentially. Conclusion is the last section.

2. Fundamental Equations in Von-Mises Coordinates

Let us introduce the vorticity function Ω and the total energy function

$$\Omega = v_x - u_y \tag{12}$$

$$L = p + \frac{1}{2}(u^2 + v^2) - \frac{2\mu u_x}{\mathrm{R_e}} \tag{13}$$

And functions A and B as follow

$$A = \mu(u_y + v_x), B = 4\mu\, u_x \tag{14}$$

Consider the allowable change of coordinates (ϕ,ψ).

$$x = x(\phi,\psi), \qquad y = y(\phi,\psi) \tag{15}$$

Where the curves $\psi = const.$ are streamline and $\phi = const.$ are arbitrary such that the Jacobian $J = \frac{\partial(x,y)}{\partial(\phi,\psi)} \neq 0$ and finite. Letting ξ be the angle between the tangents to the curves $\psi = const.$ and $\phi = const.$ at a point $P(x,y)$, streamline pattern equation (11) and applying differential geometric technique of [19] it is easy to show that the fundamental equations are following

$$-R_e\,\Omega\, J\, E = R_e\, J\sqrt{E}\left[-F\left(F_1\cos\xi + F_2\sin\xi\right) + J\left(F_1\sin\xi - F_2\cos\xi\right)\right] + R_e\, J\, E\, L_\psi + A_\phi\left((F^2 - J^2)\cos 2\xi - 2FJ\sin 2\xi\right)$$

$$+\ EA_\psi\left(J\sin 2\xi - F\cos 2\xi\right) - B_\phi\left(\frac{1}{2}(F^2-J^2)\sin 2\xi + FJ\cos 2\xi\right) + E\,B_\psi\left(\frac{1}{2}F\sin 2\xi + J\cos^2\xi\right) \quad (16)$$

$$0 = R_e J\sqrt{E}\left[F_1\cos\xi + F_2\sin\xi\right] - R_e J\,L_\phi + E\,A_\psi\cos 2\xi - A_\phi\left[F\cos 2\xi - J\sin 2\xi\right] + B_\phi\left(\frac{1}{2}F\sin 2\xi - J\sin^2\xi\right) - \frac{E\,B_\psi}{2}\sin 2\xi \quad (17)$$

And

$$\frac{1}{J\,P_{e'}}\left[\left(\frac{G\,T_\phi - F\,T_\psi}{J}\right)_\phi + \left(\frac{E\,T_\psi - F\,T_\phi}{J}\right)_\psi\right] = -\frac{\mathrm{E_c}}{\mathrm{R_e}}\,\frac{1}{4\mu}\left(B^2+4A^2\right) + \frac{T_\phi}{J} \quad (18)$$

Where E, F and G are the coefficients of first fundamental form, $J = \pm\sqrt{EG-F^2}$ and

$$B(\phi,\psi) = \frac{4\mu}{EJ^3}\left[\,E_\phi\,(F\sin\xi + J\cos\xi)^2 - 2\,E(F\sin\xi + J\cos\xi)\ (F_\phi\sin\xi + J_\phi\cos\alpha) + E^2(J_\psi\sin 2\xi + G_\phi\sin^2\xi)\,\right] \quad (19)$$

$$A(\phi,\psi) = \mu\ [-\ \frac{(F\cos\xi - J\sin\xi)}{4E^2J^5}\ \{E_\phi(2\,E\,J^3\cos\xi + F\sqrt{E}\sin\xi) - 4E^2J^2J_\phi\cos\xi - 2\,E\sqrt{E}\,F_\phi\sin\xi + E\sqrt{E}\,E_\psi\sin\xi\}$$

$$+\frac{\cos\alpha}{2J^3}\{E_\psi(F\sin\xi + J\cos\xi) - 2EJ_\psi\cos\xi - E\,G_\phi\sin\xi\} + \frac{(F\sin\xi + J\cos\xi)}{2EJ^3}\{(J\,E_\phi - 2EJ_\phi)\sin\xi + \cos\xi\ [-FE_\phi + 2\,E\,F_\phi - E\,E_\psi]\}$$

$$-\frac{\sin\xi}{2J^3}\{(E_\psi(J\sin\xi - F\cos\xi) - 2EJ_\psi\sin\xi + EG_\phi\cos\xi\}] \quad (20)$$

And

$$\Omega = \frac{(F\sin\xi + J\cos\xi)}{2EJ^3}\{(J\,E_\phi - 2EJ_\phi)\sin\xi + \cos\xi\ [-FE_\phi + 2\,E\,F_\phi - E\,E_\psi]\} - \frac{\sin\xi}{2J^3}\{E_\psi(J\sin\xi - F\cos\xi) - 2EJ_\psi\sin\xi + EG_\phi\cos\xi\}]$$

$$+\frac{(F\cos\xi - J\sin\xi)}{4E^2J^5}\{E_\phi(2\,E\,J^3\cos\xi + F\sqrt{E}\sin\xi) - 4E^2J^2J_\phi\cos\xi - 2\,E\sqrt{E}\,F_\phi\sin\xi + E\sqrt{E}\,E_\psi\sin\xi\}$$

$$-[\frac{\cos\xi}{2J^3}\{E_\psi(F\sin\xi + J\cos\xi) - 2EJ_\psi\cos\xi - E\,G_\phi\sin\xi\}] \quad (21)$$

Setting equations (6) and (8) in Martin's system we find that

$$\cos\xi = \frac{1}{\sqrt{E}} \quad (22)$$

$$E = 1 + (M + N\nu)^2 \quad (23)$$

$$F = \sqrt{1-E} \quad (24)$$

$$G = J^2 \quad (25)$$

$$J = x\,g\nu' \quad (26)$$

$$N(x) = x\,g'(x)M(x) = x\,f'(x) \quad (27)$$

The basic equations (16-21) in von-Mises coordinates on utilizing equations (22-27) becomes

$$-R_e\,\Omega = -R_e\,J\,F_2 + R_e\,L_\psi - J\,A_x + \sqrt{E-1}\,A_\psi + B_\psi \tag{28}$$

$$0 = R_e\left(F_1 + F_2\sqrt{E-1}\right) - R_e\,L_x + \frac{A_\psi(2-E)}{J} + A_x\sqrt{E-1} - \frac{\sqrt{E-1}\,B_\psi}{J} \tag{29}$$

$$\begin{aligned} &J\,T_{xx} - 2a\sqrt{E-1}\,T_{\nu x} + \frac{a^2 E}{J}T_{\nu\nu} + \left(J_x - \frac{E_\psi}{2\sqrt{E-1}} - P_{e'}\right)T_x \\ &\quad + a\left(\frac{E_\psi}{J} - \frac{E_x}{2\sqrt{E-1}} - \frac{EJ_\psi}{J^2}\right)T_\nu = -\frac{J\,E_c\,P_r}{4\mu}\left(B^2 + 4A^2\right) \end{aligned} \tag{30}$$

$$B = \frac{-4\mu(N+g)}{a\,x^2 g^2} \tag{31}$$

$$A = \frac{\mu}{a(x\,g)^2}\left[x\,g\,(M' + N'\nu) - 2(M + N\nu)(N+g)\right] \tag{32}$$

$$\Omega = \left[\frac{M'}{x\,g} - \frac{2\,M\,N}{(x\,g)^2}\right]\left(\frac{1}{a}\right) + \left[\frac{N'}{x\,g} - \frac{2N^2}{(x\,g)^2}\right]\left(\frac{\nu}{a}\right) \tag{33}$$

And the magnitude of $\mathbf{v} = (u, v)$ is

$$|\mathbf{v}| = \frac{\sqrt{1 + (M + N\nu)^2}}{J} \tag{34}$$

The condition $L_{x\psi} = L_{\psi x}$ on equations (28-29) implies

$$\begin{aligned} &JA_{xx} - 2\sqrt{E-1}\,A_{x\psi} - \frac{(2-E)}{J}A_{\psi\psi} + A_x\left(J_x - \frac{E_\psi}{2\sqrt{E-1}}\right) \\ &+ A_\psi\left(-\frac{E_x}{2\sqrt{E-1}} + \frac{J_\psi(2-E)}{J^2} + \frac{E_\psi}{J}\right) - g'(x) \neq 0 \\ &= R_e\,\Omega_x + R_e\left(F_1 + F_2\sqrt{E-1}\right)_\psi - R_e\left(J\,F_2\right)_x \end{aligned} \tag{35}$$

The equations (31-33) guide us to discus exact solution for the following two cases when $\nu''(\psi) = 0$ and when $\nu''(\psi) \neq 0$.

3. Exact Solutions

Case I: $\nu''(\psi) = 0$

For this case

$$\nu = a\psi + b \tag{36}$$

With $a \neq 0$, b as constants. Equation (35) on utilizing equation (36) provides

$$\begin{aligned} &a\,x\,g\,A_{xx} - 2x(f' + g'\nu)\,A_{x\psi} - \frac{\left[1 - x^2(f' + g'\nu)^2\right]}{a\,x\,g}A_{\psi\psi} \\ &a\,g\,A_x - A_\psi\left((f' + g'\nu) + x(f'' + g''\nu)\right) - A_\psi\left((f' + g'\nu) + x(f'' + g''\nu)\right) \\ &-\left\{B_x - \frac{(f' + g'\nu)B_\psi}{a\,g}\right\}_\psi = R_e\,\Omega_x + R_e\left(F_1 + F_2\,x(f' + g'\nu)\right)_\psi - R_e\left(a\,x\,g\,F_2\right)_x \end{aligned} \tag{37}$$

Solution of equation (37) will lead to L and T from equations (28-30), μ from equation (31) or (32), the pressure p from equation (13) and velocity components u, v from (8).

The equation (37) involves both the function A and B. The functions A and B involves components of velocity field. Therefore, one of the techniques to solve equation (37) could be to obtain a relation between the functions A and B using equations (31-32). It is easy to see that such a relation is not possible. Therefore, let us consider the following two velocity field cases

Case I(a): $A = 0$

Case I(b): $B = 0$

Case I(a):

When

$$A = 0 \tag{38}$$

The equation (27) and equation (31) imply following coupled equations

$$\frac{2(xg)'g'}{g} - (xg')' = 0 \tag{39}$$

And

$$\frac{-2(xg)'f'}{g} + (xf')' = 0 \tag{40}$$

Equations (39-40) implies

$$g(x) = \frac{-1}{(C_0 x^2 + C_1)} \tag{41}$$

$$f(x) = \frac{-n_1}{(C_0 x^2 + C_1)} + n_2 \tag{42}$$

Where $C_0 \neq 0$, C_1, n_1 and n_2 are constants. Therefore, equation (37) leads to

$$\left\{-B_x + \frac{(f' + g'v)B_\psi}{ag}\right\}_\psi = R_e\,\Omega_x + R_e\left(F_1 + F_2\,x(f' + g'v)\right)_\psi - R_e\left(a\,x\,g\;F_2\right)_x \tag{43}$$

The search for possible forms of F_1 and F_2, satisfying equation (43), equations (28-29) and equation (30) leads to

$$R_e\left(x\,g\,a\,F_2\right) = R_e\,\Omega + G_1(\psi) \tag{44}$$

And equation (43) implies

$$R_e F_1 = -(f' + g'v)\left\{\frac{R_e\,\Omega}{a\,g} + \frac{G_3(\psi)}{a\,g}\right\} - B_x + \frac{(f' + g'v)B_\psi}{a\,g} + H_1(x) \tag{45}$$

Where $G_1(\psi)$ and $H_1(x)$ are functions of integration.

Utilizing equations (44–45), in equations (28–29) and solving for the function L, we obtain

$$R_e\,L = -B + \int H_1(x)\,dx + \int G_1(\psi)\,d\psi + m_1 \tag{46}$$

Where m_1 is constant. We find viscosity from equation (32)

$$\mu = \frac{-a\,x^2 g^2}{4(N + g)} B \tag{47}$$

Equation (30), on supplying equation (38), equations (41-42) and equation (47), implies the function B

$$\begin{aligned}&(a\,x\,g)T_{xx} - 2\,a\,x(n_1\,g' + g')T_{vx} + \frac{a\left\{1 + x^2(n_1\,g' + g'v)^2\right\}}{x\,g}T_{vv} + \left(a\,g - P_{e'}\right)T_x\\ &\left(\frac{2\,x(n_1\,g' + g'v)g'}{g} - (n_1\,g' + g'v) + (n_1\,g'' + g''v)\right)a\,T_v = \frac{E_c\,P_r(N + g)}{x\,g}B\end{aligned} \tag{48}$$

Equations (47-48) provides following relation between viscosity and temperature

$$\mu = \frac{-a x^3 g^3}{4E_c P_r (N+g)^2}[(a x g) T_{xx} - 2 a x (n_1 g' + g') T_{\nu x} + \frac{a\{1 + x^2 (n_1 g' + g'\nu)^2\}}{x g} T_{\nu\nu}$$
$$+ (a g - R_e P_r) T_x + \left(\frac{2x(n_1 g' + g'\nu) g'}{g} - (n_1 g' + g'\nu) + (n_1 g'' + g''\nu) \right) a T_\nu] \tag{49}$$

One can find pressure p from equation (13) using equation (46) and can find velocity components from (8) using (41-42) in (11).

Case I(b):

Now when

$$B = 0 \tag{50}$$

Equation (32) implies

$$g = \frac{c}{x} \tag{51}$$

Where a non-zero constant is c. The equation (37) for the case (50), becomes

$$a c A_{xx} - 2a\left(M - \frac{c\nu}{x}\right) A_{x\nu} - \frac{a\left[1 - \left(M - \frac{c\nu}{x}\right)^2\right]}{c} A_{\nu\nu}$$
$$+ \frac{a c}{x} A_x - a A_\nu \left(\left(M' + \frac{c\nu}{x^2}\right) + \frac{2\left(M - \frac{c\nu}{x}\right)}{x} \right) \tag{52}$$
$$= R_e \Omega_x + a R_e \left(F_1 + \left(M - \frac{c\nu}{x}\right) F_2 \right)_\nu - R_e \left(ac F_2 \right)_x$$

The coefficients suggest searching solution of the type

$$A = R(x) + d\nu \tag{53}$$

In equation (52), where d is constant. This implies

$$\left\{ a c R'' + \frac{a c}{x} R' \right\} - a d \left(M' + \frac{2M}{x} \right) = R_e \left(\frac{1}{a c} \right) \left[M' + \frac{2M}{x} \right]' + \frac{2 R_e \nu}{a x^3}$$
$$+ a R_e \left(F_1 + (M - \frac{c\nu}{x}) F_2 \right)_\nu - R_e \left(ac F_2 \right)_x \tag{54}$$

The search for possible forms of F_1 and F_2, satisfying equation (54), equations (28-29) and equation (30) leads to

$$R_e\, ac F_2 = a d \left(M + 2\int \left(\frac{M}{x} \right) dx \right) + \left(\frac{R_e}{a c} \right) \left[M' + \frac{2M}{x} \right] - a c R' - a c \int \left(\frac{R'}{x} \right) dx + P_1(\nu) \tag{55}$$

On substituting equation (55) in equation (52), we find

$$a R_e F_1 = -\frac{1}{c}\left(M - \frac{c\nu}{x} \right) [a d \left(M + 2\int \left(\frac{M}{x} \right) dr \right) + \left(\frac{R_e}{a c} \right) \left[M' + \frac{2M}{x} \right]$$
$$- a c R' - a c \int \left(\frac{R'}{x} \right) dx + P_1(\nu)] - \left(\frac{R_e}{a x^3} \right) \nu^2 + e^x a \int M R' e^{-x} dx + E_1 e^x \tag{56}$$

Equations (28-29) on utilizing (55-56) provide

$$R_e\, a\, L=\nu\left(\frac{R_e}{a\,c}\right)\left[M'+\frac{2M}{x}\right]-\nu\, a\, c\int\left(\frac{R'}{x}\right)dx-\nu\left[M'+\frac{2M}{x}\right]\left(\frac{R_e}{ac}\right)$$
$$+\frac{R_e\,\nu^2}{2\,ax^2}+e^x\, a\int MR'\, e^{-x}\, dx+E_1\, e^x+\int P_1(\nu)\, d\nu \quad (57)$$

And

$$d=0 \quad (58)$$

Equation (31) on using equation (51) provides

$$\mu=\frac{a\,cR(x)}{(M'-g'\nu)} \quad (59)$$

In light of equations (58-59), equation (50) and equation (53) the equation (30) implies

$$a\,c\,T_{xx}-2a\left(M-\frac{c\,\nu}{x}\right)T_{\nu x}+\frac{a\left[1+\left(M-\frac{c\,\nu}{x}\right)^2\right]}{c}T_{\nu\nu}+\left(\frac{a\,c}{x}-P_{e'}\right)T_x$$
$$-\left(\frac{2\left(M+N\nu\right)}{x}+\left(M'+\frac{c\,\nu}{x^2}\right)\right)a\,T_\nu=-E_c\,P_r\left(R\,M'+\frac{c\,R}{x^2}\nu\right) \quad (60)$$

Let us searching solution of equation (60) of the form

$$T=R_1(x)+R_2(x)\nu+R_3(x)\nu^2 \quad (61)$$

Equation (60) on substituting equation (61) gives

$$\nu^2\,[\,a\,cR_3''+4agR_3'+\frac{2aR_3g^2}{c}+2aR_3\,(g'+\frac{2g}{x})+\left(\frac{ac}{x}-P_{e'}\right)R_3'\,]+\nu\,[\,a\,cR_2''-4aMR_3'+2ag\,R_2'-\frac{4aR_3M\,g}{c}+\left(\frac{ac}{x}-P_{e'}\right)R_2'$$
$$+a\left\{-2R_3\left(M'+\frac{2M}{x}\right)+R_2\left(g'+\frac{2g}{x}\right)\right\}]+\,a\,cR_1''-2aMR_2'+\frac{2aR_3(1+M^2)}{c}$$
$$+\left(\frac{ac}{x}-P_{e'}\right)R_1'-a\left(M'+\frac{2M}{x}\right)R_2=-E_c\,P_r\left(RM'+\frac{cR}{x^2}\nu\right) \quad (62)$$

The coefficients of equation (62) when compared implies

$$R_3''+\left(\frac{5}{x}-\frac{P_{e'}}{a\,c}\right)R_3'+\left(\frac{4}{x^2}\right)R_3=0 \quad (63)$$

$$R_2''+\left(\frac{3}{x}-\frac{P_{e'}}{ac}\right)R_2'+\frac{R_2}{x^2}=4aMR_3'+2aR_3\left(M'+\frac{4M}{x}\right)-E_c\,P_r\left(\frac{c}{x^2}R\right) \quad (64)$$

$$R_1''+\left(\frac{1}{x}-\frac{P_{e'}}{ac}\right)R_1'=\frac{2MR_2'}{c}+\frac{1}{c}\left(M'+\frac{2M}{x}\right)R_2-\left(\frac{2(1+M^2)}{c^2}\right)R_3-\frac{E_c\,P_r\,M'}{a\,c}R \quad (65)$$

Solving homogeneous equation (63), using the computer algebra system (CAS) software Mathematica, provides

$$R_3(x)=\frac{a^2c^2}{P_{e'}^{\,2}}\left(\frac{1}{x^2}-\frac{P_{e'}}{ac}\left(\frac{2}{x}\right)+\frac{P_{e'}^{\,2}}{2a^2c^2}\right)E_2 \\ +E_3\ \text{MeijerG}[\{\{\},\{1\}\},\{\{-2,\ 2\},\{\}\},\ \frac{-P_{e'}}{ac}x] \tag{66}$$

In order to make the equation (64) homogeneous and find its solution through Mathematica, let us set

$$R(x)=\frac{4ax^2M}{cE_c\,P_r}R_3'+\frac{2ax^2}{cE_c\,P_r}\left(M'+\frac{4M}{x}\right)R_3 \tag{67}$$

in equation (64) to find

$$R_2''+\left(\frac{3}{x}-\frac{P_{e'}}{ac}\right)R_2'+\frac{R_2}{x^2}=0 \tag{68}$$

The software Mathematica gives

$$R_2(x)=\frac{ac}{R_e\,P_r}\left(\frac{1}{x}-\frac{P_{e'}}{ac}\right)E_4+E_5\ \text{MeijerG}[\{\{\},\{1\}\},\{\{-2,\ 2\},\{\}\},\ \frac{-P_{e'}}{ac}x] \tag{69}$$

Solving equation (65), we have

$$R_1(x)=\int\left\{\frac{e^{\left(\frac{P_{e'}}{ac}\right)x}}{x}\int x\,e^{-\left(\frac{P_{e'r}}{ac}\right)x}Z_1(x)\,dx\right\}dx+E_6\int\left\{\frac{e^{\left(\frac{P_{e'}}{ac}\right)x}}{x}\right\}dx+E_7 \tag{70}$$

Where

$$Z_1(x)=\frac{2MR_2'}{c}+\frac{1}{c}\left(M'+\frac{2M}{x}\right)R_2-\left(\frac{2(1+M^2)}{c^2}\right)R_3-\frac{E_c\,P_r\,M'}{ac}R \tag{71}$$

and $E_i,\ i\in\{1,2,\cdots,7\}$ are constants.

Equation (59) using equation (67) provides viscosity, equation (61), by supplying equations (66-71), gives T, equation (13) using equation (57) gives p and equation (8) using equation (51) and equation (8) provides velocity.

Case II: $\nu''(\psi)\neq 0$

For this case let us set

$$\nu(\psi)=e^{\psi} \tag{72}$$

And

$$g=\frac{c}{x} \tag{73}$$

Because $g'(x)\neq 0$.

Equations (31-33) on utilizing equations (72-73) lead to

$$f(x)=\ln x+b \tag{74}$$

$$\Omega=\left(\frac{2}{c^2}\right)\left(\frac{1}{(e^{\psi})^2}\right) \tag{75}$$

$$A=-\frac{\mu}{c^2e^{\psi}}\,\frac{2c}{x}\left(1-\frac{ce^{\psi}}{x}\right) \tag{76}$$

And

$$B=\frac{4\mu}{c^2e^{\psi}}\left(1-\frac{ce^{\psi}}{x}\right)\left(\frac{1}{e^{\psi}}\right) \tag{77}$$

Where c and b are constant.

The equations (76-77) imply a relation between the function A and B.

$$B=\frac{-2x}{ce^{\psi}}A \tag{78}$$

Equation (35) on substituting equation (78) becomes

$$ce^{\psi}A_{xx}-2\left(1-\frac{ce^{\psi}}{x}-\frac{xe^{-\psi}}{c}\right)A_{x\psi}+\left(-\frac{2}{x}+\frac{ce^{\psi}}{x^2}+\frac{2e^{-\psi}}{c}-\frac{2xe^{-2\psi}}{c^2}\right)A_{\psi\psi}+\left(\frac{ce^{\psi}}{x}-\frac{2xe^{-\psi}}{c}\right)A_x+\left[\frac{6xe^{-2\psi}}{c^2}-\frac{2e^{-\psi}}{c}\right]A_{\psi}-\frac{4xe^{-2\psi}}{c^2}A$$

$$= R_e\left(F_1\right)_\psi + R_e\left((1-\frac{ce^\psi}{x})F_2\right)_\psi - R_e\left(ce^\psi F_2\right)_x \tag{79}$$

The search for possible forms of F_1 and F_2, satisfying equation (79), equations (28-29) and equation (30) leads to

$$R_e\left((1-\frac{ce^\psi}{x})F_2\right)_\psi - R_e\left(ce^\psi F_2\right)_x = 0 \tag{80}$$

Or

$$R_e\, F_2 = \frac{R_e}{x}\, H\left(\frac{e^\psi}{x}+\frac{1}{c}\ln x\right) \tag{81}$$

Where $H\left(\frac{e^\psi}{x}+\frac{1}{c}\ln x\right)$ is an arbitrary function. The equation (79) on substituting equation (81) provides

$$R_e\left(F_1\right)_\psi = c\,e^\psi A_{xx} - 2\left(1-\frac{ce^\psi}{x}-\frac{x\,e^{-\psi}}{c}\right)A_{x\psi} + \left(-\frac{2}{x}+\frac{ce^\psi}{x^2}+\frac{2e^{-\psi}}{c}-\frac{2\,x\,e^{-2\psi}}{c^2}\right)A_{\psi\psi}$$

$$+\left(\frac{ce^\psi}{x}-\frac{2\,x\,e^{-\psi}}{c}\right)A_x + \left[\frac{6\,x\,e^{-2\psi}}{c^2}-\frac{2e^{-\psi}}{c}\right]A_\psi - \frac{4\,x\,e^{-2\psi}}{c^2}A \tag{82}$$

The integration of equation (82) requires selecting the form of A . Let

$$A(x,\psi) = M(x)\,S(\psi) \tag{83}$$

The equation (82) on substituting (83) gives

$$R_eF_1 = c\,M''\int e^\psi\, S d\psi - \frac{c\,M'}{x}\int e^\psi\, S\, d\psi + \frac{c\,M}{x^2}\int e^\psi\, S\, d\psi + \left(-2+\frac{2c}{x}e^\psi+\frac{2\,x}{c}e^{-\psi}\right)M'\, S$$

$$+\left(-\frac{2}{x}-\frac{c}{x^2}e^\psi+\frac{2\,x}{c^2}e^{-2\psi}\right)M\, S + \left(\frac{c}{x^2}e^\psi+\frac{2}{c}e^{-\psi}-\frac{2\,x}{c^2}e^{-2\psi}\right)M\, S' + K(x) \tag{84}$$

Where $K(x)$ is a function of integration. It is easy to see that F_1 and F_2 satisfies equation (79) and the viscosity is

$$\mu = -\frac{ce^\psi\, x}{2}\left(1-\frac{ce^\psi}{x}\right)^{-1} M(x)\,S(\psi) \tag{85}$$

Equations (28-29) provides

$$R_e\, L = \left(\frac{R_e}{c^2}\right)e^{-2\psi} - \frac{2\,b_1\,x}{c}M(x)\,e^{-\psi} + \left(\frac{R_e\ln x}{x} - b_1\,c\,M'(x)\right)e^\psi + \frac{c\,R_e\,e^{2\psi}}{2\,x^2}$$

$$+b_1\,M + 2\,b_1\int\frac{M(x)}{x}dx + \frac{R_e\left(\ln x\right)^2}{2\,c} + b_2 + \int K(x)\,dx \tag{86}$$

And

$$H\left(\frac{e^\psi}{x}+\frac{1}{c}\ln x\right) = \left(\frac{e^\psi}{x}+\frac{1}{c}\ln x\right) \tag{87}$$

Where b_1 and b_2 are constants.

Equation (30) on utilizing equation (72-74) and equation (85) simplifies to

$$c e^{\psi} T_{xx} - 2\left(1 - \frac{c e^{\psi}}{x}\right) T_{\psi x} + \left(\frac{2e^{-\psi}}{c} - \frac{2}{x} + \frac{c e^{\psi}}{x^2}\right) T_{\psi\psi} + \left(\frac{c e^{\psi}}{x} - P_{e'}\right) T_x - \left(\frac{2e^{-\psi}}{c}\right) T_{\psi} =$$

$$-2 b_1 E_c P_r \left(\frac{x e^{-2\psi}}{c^2} - \frac{e^{-\psi}}{c} + \frac{1}{x} - \frac{c e^{\psi}}{x^2}\right) M(x) \quad (88)$$

The right-hand side of equation (88) suggests searching for solution of the form

$$T(x,\psi) = K_1(x) + K_2(x) e^{\psi} + K_3(x) e^{-\psi} + K_4(x) e^{-2\psi} \quad (89)$$

Equation (88) on substituting equation (89) reduces to

$$\frac{12 K_4 e^{-3\psi}}{c} + \left[4K_4' - \frac{8K_4}{x} - P_{e'} K_4' + \frac{4K_3}{c}\right] e^{-2\psi}$$

$$+ \left[c K_4'' - \frac{4c K_4'}{x} + \frac{4c K_4}{x^2} + \frac{c K_4'}{x} + (2 - P_{e'}) K_3' - \frac{2 K_3 e^{-\psi}}{x}\right] e^{-\psi} + c K_3'' - \frac{c K_3'}{x} + \frac{c K_3}{x^2} - P_{e'} K_1'$$

$$+ \left[c K_1'' - (2 + P_{e'}) K_2' - \frac{2 K_2}{x} + \frac{c K_1'}{x}\right] e^{\psi} + \left[c K_2'' + \frac{3c K_2'}{x} + \frac{c K_2}{x^2}\right] e^{2\psi} = -2 b_1 E_c P_r \left(\frac{1}{x} - \frac{c e^{\psi}}{x^2} - \frac{e^{-\psi}}{c} + \frac{x e^{-2\psi}}{c^2}\right) M(x) \quad (90)$$

The equation (90) implies

$$M = t_0 x^{\alpha} \text{ where } \alpha = \frac{(-4 + P_{e'})}{(2 - P_{e'})}, \text{ when } P_{e'} \neq 2 \quad (91)$$

$$K_1 = \frac{E_c b_1}{2R_e}\left\{-x M' + 2\int \left(\frac{M}{x}\right) dx\right\} + b_3 \quad (92)$$

$$K_2 = \frac{b_4}{x} \quad (93)$$

$$K_3 = -\frac{E_c P_r b_1}{2c} x M \quad (94)$$

$$K_4 = 0 \quad (95)$$

Where b_3, b_4 and t_0 are arbitrary constants and

$$P_{e'} C_1 - \frac{c E_c m_1 t_0}{2R_e} x^{\alpha} \left[\alpha^3 - 4\alpha + 4 P_{e'}\right] = 0 \quad (96)$$

The solution equation (96) depends on the choice of α. It will be discussed for $\alpha = 0$ and $\alpha \neq 0$.

For the case $\alpha = 0$ (or $P_{e'} = 4$), the equations (46-50) simplifies to

$$M = t_0 x^{\alpha} = t_0, \; b_4 = \frac{2 c E_c b_1 t_0}{R_e}, \; K_1(x) = \frac{2 E_c b_1 t_0 \ln x}{R_e} + b_3,$$

$$K_2(x) = \frac{2 c E_c b_1 t_0}{R_e} \frac{1}{x}, \; K_3 = -\frac{E_c P_r b_1 t_0}{2c} x \text{ and } K_4 = 0$$

Implies

$$T = b_3 + \frac{E_c b_1 t_0 P_r}{2}\left\{\ln x + \frac{c e^{\psi}}{x} - \frac{x}{c e^{\psi}}\right\} \quad (97)$$

For the case $\alpha \neq 0$ (or $P_{e'} \neq 4$) the equations (37-41) simplifies to

$$b_4 = 0, \; b_1 = 0, \; M = t_0 x^{\alpha}$$

$$K_1(x) = b_3, \; K_2(x) = 0, \; K_3 = 0, \text{ and } K_4 = 0$$

Implies

$$T = b_2 \quad (98)$$

It finds viscosity from (84), pressure from (11) using (86) and velocity from equation (8). The temperature T is found form equation (97) for $P_{e'} = 4$ otherwise T satisfies equation (98).

4. Conclusion

A class of exact solutions for plane steady motion of incompressible fluids of variable viscosity in the presence of body force with moderate Peclet number is obtained. The non-dimensional form of basic equations undergoes the successive transformations until equations in von-Mises coordinates. Two classes of streamline pattern $y = f(x) + g(x)\nu(\psi)$ are considered with $\nu(\psi) = a\psi + b$ and $\nu(\psi) = e^{\psi}$. When the family of streamlines varies linearly, exact solutions for a suitable component of body force are determined based on two velocity field cases. The first velocity field fixes both of the functions $f(x) = n_1 g(x) + n_2$ and $g(x) = \frac{-1}{C_0 x^2 + C_1}$. The second velocity field provide exact solution for arbitrary $f(x)$ and requires $g(x) = \frac{c}{x}$. Where $c \neq 0$, $C_0 \neq 0$, C_1, n_1 and n_2 are constants. When the family of streamlines varies exponentially, the streamline pattern is $y = b + \ln x + \frac{c\, e^{\psi}}{x}$. The temperature distribution, due to heat generation, is $T = b_3 + \frac{E_c b_1 t_0 P_r}{2}\left\{\ln x + \frac{c\, e^{\psi}}{x} - \frac{x}{c\, e^{\psi}}\right\}$ when $P_{e'} = 4$. Whenever $P_{e'} \neq 4$, the temperature distribution is constant. There are infinite set of expressions for streamlines, pressure, viscosity, temperature distribution and velocity vector field in the presence of body force.

The software Mathematica is used to determine the solution of some ordinary differential equations. Using Mathematica one can draw the streamlines pattern to observe the effect of various parameters on the streamlines and discuss the flow characteristic.

References

[1] Naeem, R. K.; Exact solutions of flow equations of an incompressible fluid of variable viscosity via one – parameter group: *The Arabian Journal for Science and Engineering*, 1994, 19 (1), 111-114.

[2] Naeem, R. K.; Srfaraz, A. N.; Study of steady plane flows of an incompressible fluid of variable viscosity using Martin's System: *Journal of Applied Mechanics and Engineering*, 1996, 1 (1), 397-433.

[3] Naeem, R. K.; Steady plane flows of an incompressible fluid of variable viscosity via Hodograph transformation method: *Karachi University Journal of Sciences*, 2003, 3 (1), 73-89.

[4] Naeem, R. K.; On plane flows of an incompressible fluid of variable viscosity: *Quarterly Science Vision*, 2007, 12 (1), 125-131.

[5] Naeem, R. K.; Mushtaq A.; A class of exact solutions to the fundamental equations for plane steady incompressible and variable viscosity fluid in the absence of body force: *International Journal of Basic and Applied Sciences*, 2015, 4 (4), 429-465. www.sciencepubco.com/index.php/IJBAS, doi: 10.14419/ijbas.v4i4.5064.

[6] Gerbeau, J. -F.; Le Bris, C., A basic Remark on Some Navier-Stokes Equations With Body Forces: *Applied Mathematics Letters*, 2000, 13 (1), 107-112.

[7] Giga, Y.; Inui, K.; Mahalov; Matasui S.; Uniform local solvability for the Navier-Stokes equations with the Coriolis force: *Method and application of Analysis*, 2005, 12, 381-384.

[8] Landau L. D. and Lifshitz E. M.; Fluid Mechanics, Pergmaon Press, vol 6.

[9] Mushtaq A., On Some Thermally Conducting Fluids: Ph. D Thesis, Department of Mathematics, University of Karachi, Pakistan, 2016.

[10] Mushtaq A.; Naeem R. K.; S. Anwer Ali; A class of new exact solutions of Navier-Stokes equations with body force for viscous incompressible fluid,: *International Journal of Applied Mathematical Research*, 2018, 7 (1), 22-26. www.sciencepubco.com/index.php/IJAMR, doi: 10.14419/ijamr.v7i1.8836.

[11] Mushtaq Ahmed, Waseem Ahmed Khan,: A Class of New Exact Solutions of the System ofPDEfor the plane motion of viscous incompressible fluids in the presence of body force,: *International Journal of Applied Mathematical Research*, 2018, 7 (2), 42-48. www.sciencepubco.com/index.php/IJAMR, doi: 10.14419/ijamr.v7i2.9694.

[12] Mushtaq Ahmed, Waseem Ahmed Khan, S. M. Shad Ahsen: A Class of Exact Solutions of Equations for Plane Steady Motion of Incompressible Fluids of Variable viscosity in presence of Body Force: *International Journal of Applied Mathematical Research*, 2018, 7 (3), 77-81. www.sciencepubco.com/index.php/IJAMR, doi: 10.14419/ijamr.v7i2.12326.

[13] Mushtaq Ahmed, (2018), A Class of New Exact Solution of equations for Motion of Variable Viscosity Fluid In presence of Body Force with Moderate Peclet number, *International Journal of Fluid Mechanics and Thermal Sciences,* 4 (4)429- www.sciencepublishingdroup.com/j/ijfmts doi: 10.11648/j.ijfmts.20180401.12.

[14] D. L. R. Oliver & K. J. De Witt, High Peclet number heat transfer from adroplet suspended in an electric field: Interior problem, *Int. J. Heat Mass Transfer*, vol. 36: 3153-3155, 1993.

[15] Z. G. Feng, E. E. Michaelides, Unsteady heat transfer from a spherical particle atfinite Peclet numbers, *J. Fluids Eng.* 118: 96-102, 1996.

[16] Fayerweather Carl, Heat Transfer From a Droplet at Moderate Peclet Numbers with heat Generation. *PhD. Thesis, U of Toledo*, May 2007.

[17] Martin, M. H.; The flow of a viscous fluid I: *Archive for Rational Mechanics and Analysis*, 1971, 41 (4), 266-286.

[18] Daniel Zwillinger; Handbook of differential equations; *Academic Press, Inc.* (1989).

[19] Weatherburn C. E., Differential geometry of three Dimensions, *Cambridge University Press*, (1964).